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ON THE HOMOLOGICAL FINITENESS PROPERTIES OF SOME MODULES OVER METABELIAN LIE ALGEBRAS

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ABSTRACT

We characterise the modules B of homological type FP_m over a finitely generated Lie algebra L such that L is a split extension of an abelian ideal A and an abelian subalgebra Q and A acts trivially on B. The characterisation is in terms of the invariant Δ introduced by R. Bryant and J. Groves and is a Lie algebra version of the generalisation [K 4, Conjecture 1] of the still open FP_m-Conjecture for metabelian groups [Bi-G, Conjecture p. 367]. The case $m = 1$ of our main result is treated separately, as there the characterisation is proved without restrictions on the type of the extension.

Introduction

The purpose of this paper is to formulate and establish in the split extension case the counterpart of the generalised FP_m-Conjecture suggested in [K 4, Conj. 1] and [K 2, Conj. 6] for finitely generated metabelian Lie algebras. The original FP_m-Conjecture [B-G 1] describes when a finitely generated metabelian group G is of homological type FP_m in terms of the invariant

$$\Sigma^1(G) \subseteq S(G) = \{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\}\}.$$

Though the FP_m-Conjecture for metabelian groups is still open it is known to hold in the following cases: $m = 2$ [B-S], $m = 3$ and G a split extension of abelian groups [B-H], G of finite Prufer rank [Å], G a torsion analogue of a group of finite

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Prufer rank [K 1]. A proof of the pro-p version of the FP_m-Conjecture for finitely generated metabelian pro-p groups suggested in [King] can be found in [K 3].

The question of finite presentability of metabelian Lie algebras is addressed in [B-G 1] and [B-G 2] where R. Bryant and J. Groves give a characterization of finite presentability in terms of the invariant Δ. In [W] links between finite presentability and an HNN-construction for Lie algebras are investigated. This approach gives the surprising result that for a finitely presented Lie algebra without free subalgebras of rank two the ideals of codimension one are finitely generated as subalgebras.

In this paper we examine some homological finiteness properties of modules over metabelian Lie algebras. A module over a Lie algebra L (over a field K) is a module over its universal enveloping algebra $U(L)$. We are primarily interested in modules B over L such that some abelian ideal A of L with L/A abelian has the property that A acts trivially on B. This includes the case of the trivial module K.

THEOREM A: *Suppose L is a finitely generated Lie algebra over a field K, A is an abelian ideal in L with $Q = L/A$ abelian and B is a finitely generated (right) module over the universal algebra $U(Q)$ of Q. Then the following are equivalent:*

1. *B is finitely presented as a module over L (i.e., as a module over the universal algebra $U(L)$ of L) where the action of L is via the canonical projection $\pi: L \to Q$.*
2. *$A \otimes_K B$ is finitely generated over the universal algebra $U(Q)$, where $U(Q)$ acts via the diagonal homomorphism $\partial: U(Q) \to U(Q) \otimes U(Q)$ sending $q \in Q$ to $q \otimes 1 + 1 \otimes q$.*
3. $\Delta(Q, A) \cap -\Delta(Q, B) = [0]$.

COROLLARY B: *Suppose L is a finitely generated Lie algebra over a field with an abelian ideal A such that $Q = L/A$ is abelian. Then L is finitely presented as a Lie algebra if and only if A is finitely presented as a module over $U(L)$.*

The group counterpart of the equivalence of conditions 1 and 3 from Theorem A is considered in [K 2, Prop. 5]. There only the case of split extension groups is solved leaving the question for non-split groups open.

The main result of this paper is the proof of the Lie algebra version of the generalised FP_m-Conjecture. In the Lie algebra case the Bryant–Groves invariant Δ will play the role of the Bieri–Strebel invariant $\Sigma^1(G)^c$. Note we establish the result only for split extensions metabelian Lie algebras. The group theoretic analogue of the equivalence of conditions 1 and 3 from Theorem C is still an open problem.

THEOREM C: *Suppose L is a finitely generated Lie algebra over a field K, A is an abelian ideal in L with $Q = L/A$ abelian and B is a finitely generated (right) module over the universal algebra $U(Q)$ of Q. We further assume that L is a split extension of A by Q. Then the following are equivalent:*

1. *B is of type FP_m over L;*
2. *$B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action;*
3. *whenever $[v_2], \ldots, [v_{m+1}] \in \Delta(Q, A), [v_1] \in \Delta(Q, B)$ and $[0] = [v_1] + \cdots + [v_{m+1}]$ then all $[v_i] = [0]$.*

1. Preliminaries on the invariant Δ

The classification of the finitely presented Lie algebras over a field K given in [B-G 1], [B-G 2] depends on the invariant $\Delta(Q, A)$, where A is an abelian ideal of L with abelian quotient $Q = L/A$. Let $K[Q]$ be the polynomial algebra on n commuting variables where n is the dimension of Q. Note that by the Poincare–Birkhoff–Witt Theorem, for every abelian Lie algebra L_0 over a field K the universal enveloping algebra $U(L_0)$ of L_0 is isomorphic to the symmetric tensor algebra $S(L_0)$ of L_0, i.e., $S(L_0)$ is the quotient of the tensor algebra of L_0 over K through the ideal generated by $x \otimes y - y \otimes x$ for $x, y \in L_0$. In particular, $K[Q]$ is isomorphic to the universal enveloping algebra $U(Q)$ of Q. By definition for a finitely generated $K[Q]$-module A

$$\Delta(Q, A) = \{[\chi] \mid \chi \in \mathrm{Hom}_K(Q, \overline{K}((t))), \chi \text{ is extendable to a homomorphism of rings } \chi' \colon K[Q]/\mathrm{Ann}(A) \to \overline{K}((t))\} \subset \mathrm{Hom}_K(Q, \overline{K}((t)))/\mathrm{Hom}_K(Q, \overline{K}[[t]]),$$

where $[\chi] = \chi + \mathrm{Hom}(Q, \overline{K}[[t]])$, $\overline{K}$ is the algebraic closure of K, $\overline{K}((t))$ is the field of fractions of $\overline{K}[[t]]$ and $\mathrm{Ann}(A)$ is the annihilator of A in $K[Q]$. The main result of [B-G 1], [B-G 2] asserts that L is finitely presented as a Lie algebra if and only if the exterior square of A (over K) is finitely generated over $K[Q]$ via the diagonal adjoint action if and only if whenever $[\chi_1], [\chi_2] \in \Delta(Q, A) \setminus \{[0]\}$ we have $[\chi_1] + [\chi_2] \neq [0]$, i.e., $\Delta(Q, A)$ has no non-zero antipodal elements. The finite generation (over $K[Q]$) of the exterior square of A turns equivalent to the finite generation (over $K[Q]$) of the tensor square of A.

LEMMA 1: *Suppose $0 \to V_1 \to V \to V_2 \to 0$ is short exact sequence of finitely generated $K[Q]$-modules. Then*

$$\Delta(Q, V) = \Delta(Q, V_1) \cup \Delta(Q, V_2).$$

Proof: Observe firstly that $\Delta(Q, V)$ is defined in terms of the annihilator of V. More precisely, it is defined in terms of the prime ideals containing the annihilator, since the kernel of the extension χ' of χ for $[\chi] \in \Delta(Q,V)$ is a prime ideal. Thus, if $\mathrm{Ann}(V) \subseteq \mathrm{Ann}(W)$, then $\Delta(Q,W) \subseteq \Delta(Q,V)$. More precisely, if $\chi'\colon \overline{K}[Q]/\operatorname{ann}(W) \to \overline{K}((t))$ is the map showing that $[\chi] \in \Delta(Q,W)$, then we can combine this with the natural surjection $\overline{K}[Q]/\operatorname{ann}(V) \to \overline{K}[Q]/\operatorname{ann}(W)$ to show that $[\chi] \in \Delta(Q,V)$. Since the annihilator of any proper submodule or quotient contains the annihilator of V, this shows that $\Delta(Q,V_1) \cup \Delta(Q,V_2) \subseteq \Delta(Q,V)$.

For the converse, suppose that I_1 and I_2 are the annihilators of V_1 and V_2. Then $I_1 I_2$ annihilates V. Suppose that $[\chi] \in \Delta(Q,V)$ and that the corresponding map induced on $\overline{K}[Q]$ is χ' with kernel P. Then $I_1 I_2 \subseteq \mathrm{Ann}(V) \subseteq P$ and so, as P is a prime ideal, $I_1 \subseteq P$ or $I_2 \subseteq P$. It follows, as in the previous paragraph, that $[\chi] \in \Delta(Q,V_1)$ or $[\chi] \in \Delta(Q,V_2)$.

Lemma 2: *Suppose V is a finitely generated $K[Q]$-module. If*

$$[\chi] \in \Delta(Q,V) \smallsetminus \{[0]\},$$

then there exists a non-trivial linear map $w\colon V \to \overline{K}((t))$ such that $w(vq) = w(v)\chi(q)$ for all $v \in V$ and $q \in Q$.

Proof: We start with the observation that in case V is 1-generated as a module over $K[Q]$ we have $V \simeq K[Q]/\operatorname{ann}(V)$ and the linear map w can be defined as the ring homomorphism $\chi'\colon K[Q] \to \overline{K}((t))$ extending χ.

Note that V is a finitely generated over $K[Q]$ and so Noetherian module. We shall assume that the statement is true for every proper quotient of V but not for V itself. Choose $[\chi] \in \Delta(Q,V)$ so that there is no associated w. Denote the kernel of χ' by P. Observe that if $[\chi] \in \Delta(Q,W)$ for any proper quotient W, then the assumption allows us to find a map $w\colon W \to \overline{K}((t))$ which can then be extended to V via the natural epimorphism $V \to W$. Thus by Lemma 1, $[\chi] \in \Delta(Q,L)$ for any non-zero submodule L of V. In particular, looking at the associated primes of V and their victims, any associated prime of V must lie in P.

Now we consider the primary decomposition $0 = \bigcap_{i \le s} L_i$ of the trivial submodule of V given by [Bo, Ch. 4, Section 2, Thm 1], i.e., all quotients V/L_i are $K[Q]$-modules with one associated prime depending on i. If V has more than one associated prime, then all L_i are non-zero, and hence V/L_i are proper quotients of V. Note V embeds in $V/L_1 \oplus \cdots \oplus V/L_s$ and by the previous lemma $[\chi] \in \Delta(Q,V) \subseteq \Delta(Q, V/L_1 \oplus \cdots \oplus V/L_s) = \bigcup_i \Delta(Q, V/L_i)$, i.e., for some i we have $[\chi] \in \Delta(V/L_i)$, a contradiction. Then we can assume that V has just one associated prime I (remember every associated prime is in P, hence $I \subseteq P$).

This implies that $V.I^m = 0$ for some m and $V/V.I$ has annihilator I in $K[Q]$. If $V.I \neq 0$, then $V/V.I$ is a proper quotient of V with annihilator I and hence $[\chi] \in \Delta(Q, V/V.I)$, a contradiction. Then we can assume that $V.I = 0$, i.e., I is the annihilator of V in $K[Q]$. Furthermore, it is well known [Bo, Ch. 4, Section 1, Prop. 2] that the maximal elements in the set $\{\mathrm{Ann}(x) \mid x \in V \smallsetminus 0\}$ are associated primes for V, thus for every $x \in V \smallsetminus \{0\}$ its annihilator in $K[Q]$ is the ideal I. All this shows that V is torsion-free as $K[Q]/I$-module.

We show that the annihilator of $V/V.P$ in $K[Q]$ is P. This will imply that if $I \neq P$ the module $V/V.P$ is a proper quotient of V and $[\chi] \in \Delta(Q, V/V.P)$, a contradiction. Suppose that the annihilator of $V/V.P$ is larger than P, i.e., contains an element $\lambda \in K[Q] \smallsetminus P$. Then for a finite generating set of $e_1, \ldots, e_m$ of V over $K[Q]$ there exist elements $f_{i,j} \in P$ such that $e_i\lambda = \sum_j e_j f_{i,j}$ for all $i \leq m$. Then $e_i \det(A) = 0$ for every $i \leq m$ where A is the matrix with entries $\lambda\delta_{i,j} - f_{i,j}$, $\delta_{i,j}$ is the Kroneker's symbol, and hence $\det(A) \in \mathrm{ann}(e_j) = I \subseteq P$. As $\det(A) \in \lambda^m + P$ we have that $\lambda^m \in P$, a contradiction.

From now on we assume $I = P$. We claim that there is a suitable map $w\colon V \to \overline{K}((t))$ extending χ, which will be a contradiction completing the proof. Set $R = K[Q]/P$ and let S denote the field of fractions of R. Observe that χ induces a homomorphism χ' from S to $\overline{K}((t))$. Also observe that R is finitely generated over K and so S has finite transcendence degree over K. As V is R-torsion-free, it embeds in $V \otimes_R S$ which is a finite dimensional vector space over S. But $\overline{K}((t))$ has infinite transcendence degree over $\overline{K}$ and so also over $\chi'(S)$. Thus $\overline{K}((t))$ contains $\chi'(S)$-subspaces of arbitrary finite dimension. Hence there is a subring of $\overline{K}((t))$ which is isomorphic to $V \otimes_R S$ via an isomorphism χ'' which agrees with χ' on S. The required map w is then given by a combination of $V \to V \otimes_R S \to \overline{K}((t))$. This completes the proof.

2. Proof of Proposition 3

This section is devoted to the proof of one of the implications of Theorem A. Our proof uses the techniques developed in [B-G 1, section 2]. As the proof is very long and technical it is split into several steps.

PROPOSITION 3: *With the assumptions of Theorem A, 2. implies 1.*

Proof: 1. Let $a_1, \ldots, a_{s_0}, y_1, \ldots, y_n$ be a generating set of L such that $a_1, \ldots, a_{s_0} \in A$ and the images $x_1, \ldots, x_n$ of $y_1, \ldots, y_n$ in $Q = L/A$ form a basis of Q. Furthermore, for all $1 \leq j < i \leq s_0$ assume $a_{i,j} = [y_i, y_j] \in \{a_1, \ldots, a_{s_0}\} \cup \{0\}$. Let F be the free Lie algebra on the generators $X_1, \ldots, X_n$ and $U(F)$ be its universal algebra. We define

$$\rho\colon U(F) \to U(Q)$$

to be the homomorphism of K-algebras sending X_i to x_i,

$$\nu\colon U(Q) \to U(F)$$

the linear map sending $x_{i_1} \cdots x_{i_k}$ to $X_{i_1} \cdots X_{i_k}$ for $i_1 \le \cdots \le i_k$ and

$$\varphi\colon U(F) \to U(L)$$

the K-algebra homomorphism sending X_i to y_i. Then $\rho = \tau \circ \varphi$, where

$$\tau\colon U(L) \to U(Q)$$

is the homomorphism of associative K-algebras induced by the canonical projection $L \to Q$.

The elements $X_{i_1}^{\alpha_1} \cdots X_{i_k}^{\alpha_k}$ and $x_1^{\beta_1} \cdots x_n^{\beta_n}$ of $U(F)$ and $U(Q)$ are called monomials of degree $\alpha_1 + \cdots + \alpha_k$ and $\beta_1 + \cdots + \beta_n$ respectively. If $f = f_1 \otimes f_2$ is a monomial in $U(F) \otimes U(F)$ (resp. $U(Q) \otimes U(Q)$) the degree of f is $\deg(f_1) + \deg(f_2)$. For a general element f of $U(F)$, $U(Q)$, $U(F) \otimes U(F)$ or $U(Q) \otimes U(Q)$ the degree $\deg(f)$ is the maximal degree of the monomials in the support of f. By definition, for a subspace J of $U(F), U(Q)$, $U(F) \otimes U(F)$ or $U(Q) \otimes U(Q)$ the subspace J_t is spanned by all elements of J of degree at most t.

Note that $U(L)$ acts on A via the adjoint (right) action. As A is abelian this makes A a right $U(Q)$-module. More precisely, if $f = gx_i$ is a monomial in $U(Q)$ the image of $a \in A$ under the action of f denoted by $a \circ f$ is $(a \circ g) \circ x_i = [a \circ g, y_i]$; furthermore, $a \circ 1 = a$ and this definition is extended by linearity for arbitrary elements of $U(Q)$. If $f \in U(L)$ we write $a \circ f$ for $a \circ \tau(f)$.

2. We adopt the notations from [B-G 1] and for an element $\lambda \in U(Q)$ write $\lambda(u)$, $\lambda(v)$ and $\lambda(d)$ for $\lambda \otimes 1, 1 \otimes \lambda \in U(Q) \otimes U(Q)$ and the image of λ under the diagonal ring homomorphism

$$\delta\colon U(Q) \to U(Q) \otimes U(Q)$$

sending $q \in Q$ to $q \otimes 1 + 1 \otimes q$. Similarly, we define for an element $\lambda \in U(F)$ elements $\lambda(U), \lambda(V)$ and $\lambda(D)$ in $U(F) \otimes U(F)$.

Now let $b_1, \ldots, b_m$ be a generating set of B over $U(Q)$. We remind the reader that $a_1, \ldots, a_{s_0}$ is a generating set of A as a $U(Q)$-module. Since $U(Q)$ is a

Noetherian ring the annihilator ideals $\mathrm{Ann}_{U(Q)} b_i$ and $\mathrm{Ann}_{U(Q)} a_j$ are finitely generated over $U(Q)$, i.e., for some elements $g_{it}, \widetilde{g}_{jt} \in U(Q)$

$$\mathrm{Ann}_{U(Q)} b_i = \sum_{t\geq 1} g_{it} U(Q), \quad \mathrm{Ann}_{U(Q)} a_j = \sum_{t\geq 1} \widetilde{g}_{jt} U(Q).$$

Similarly to [B-G 1, (3.1)]

$$(1) \qquad \mathrm{Ann}_{U(Q)\otimes U(Q)}(b_i \otimes a_j) = \mathrm{Ann}_{U(Q)}(b_i) \otimes U(Q) + U(Q) \otimes \mathrm{Ann}_{U(Q)}(a_j).$$

We claim that for every $1 \leq r \leq m, 1 \leq s \leq s_0, 1 \leq k \leq n$ there exist elements $\phi_{rskj}, \psi_{rskj} \in U(Q) \otimes U(Q), f_{rski} \in U(Q)$ and an integer l independent of r, s and k such that

$$(2) \qquad x_k(v)^{l+1} + \sum_{0\leq i\leq l} x_k(v)^i f_{rski}(d) + \sum_{j\geq 1} g_{rj}(u)\phi_{rskj} + \sum_{j\geq 1} \widetilde{g}_{sj}(v)\psi_{rskj} = 0.$$

In the case when $B = A$, formula (2) is proved in [B-G 1, (3.3)]. The general case can be proved using the same argument. For completeness we sketch a proof. The $U(Q)$-submodule of $B \otimes A$ generated by $\{b_r \otimes (a_s \circ x_k^j)\}_{j\geq 0}$ is finitely generated, say by $\{b_r \otimes (a_s \circ x_k^j)\}_{0\leq j\leq l}$. Then for some $f_{rski}(d) \in U(Q) \otimes U(Q)$,

$$x_k(v)^{l+1} + \sum_{i\leq l} x_k(v)^i f_{rski}(d) \in \mathrm{Ann}_{U(Q)\otimes U(Q)}(b_r \otimes a_s).$$

Now (2) follows immediately from (1).

3. Let $\partial\colon \bigoplus_{i\leq m} e_i U(L) \to B$ be the homomorphism of $U(L)$-modules sending the generator e_i of the free module $\bigoplus_{i\leq m} e_i U(L)$ to b_i. Then B is finitely presented over $U(L)$ if and only if $\mathrm{Ker}\,\partial$ is finitely generated over $U(L)$. We remind the reader that $\mathrm{Ann}_{U(Q)} b_i = \sum_{t\geq 1} g_{it} U(Q)$ and define

$$\widetilde{X} = \{e_i \varphi\nu(g_{ij})\}_{i,j\geq 1} \text{ and } X_t = \{e_i\varphi(f_1)(a_j \circ \rho(f_2)) \mid$$
$$f_1, f_2 \text{ monomials in } U(F), \deg(f_1 f_2) \leq t, i \leq m, j \leq s_0\}.$$

Denote by V_t the $U(L)$-submodule of $\mathrm{Ker}\,\partial \subseteq \bigoplus_{i\leq m} e_i U(L)$ generated by the finite set $X_t \cup \widetilde{X}$. We aim to prove that for sufficiently big t,

$$V_t = V_{t+1}.$$

Suppose we have done this. Then as every V_m is finitely generated over $U(L)$ the union $V = \bigcup_{m\geq 1} V_m$ is finitely generated over $U(L)$. By construction $\mathrm{Ker}\,\partial/V$ is a surjective image of the quotient of $\mathrm{Ker}\,\partial$ through the $U(L)$-submodule T

generated by $\bigcup_{t\geq 1} X_t$. Note that $\operatorname{Ker}\partial/T$ is the kernel of the homomorphism of $U(Q)$-modules $\bigoplus_{i\leq m} e_i U(Q) \to B$ sending e_i to b_i. As the domain of this homomorphism is finitely generated over $U(Q)$ and $U(Q)$ is Noetherian, we deduce that $\operatorname{Ker}\partial/T$ is finitely generated over $U(Q)$ and hence finitely generated over $U(L)$. In particular, $\operatorname{Ker}\partial/V$ is finitely generated over $U(L)$. Finally, as V is finitely generated over $U(L)$ we deduce that $\operatorname{Ker}\partial$ is finitely generated over $U(L)$, as required.

LEMMA 3.1: *If f_1, f_2, f_3 are monomials in $U(F)$ such that $\deg(f_1f_2f_3) < 2t$, then*

$$e_i\varphi(f_1)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3)) \in V_t.$$

Proof: We induct on $\deg(f_1)$. If $f_1 = 1$ then $\deg(f_2) < t$ or $\deg(f_3) < t$, say $\deg(f_3) < t$. Then $e_i(a_k \circ \rho(f_3))$ and consequently $e_i(a_k \circ \rho(f_3))(a_j \circ \rho(f_2))$ are elements of V_t.

If $f_1 = gY$ for some $Y = X_j$ we have

$$\begin{aligned} e_i\varphi(f_1)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3)) &= e_i\varphi(g)[\varphi(Y), a_j \circ \rho(f_2)](a_k \circ \rho(f_3)) \\ +e_i\varphi(g)(a_j \circ \rho(f_2))&[\varphi(Y), a_k \circ \rho(f_3)] + e_i\varphi(g)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3))\varphi(Y) \\ = -e_i\varphi(g)(a_j \circ \rho(f_2Y))&(a_k \circ \rho(f_3)) - e_i\varphi(g)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3Y)) \\ &+e_i\varphi(g)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3))\varphi(Y). \end{aligned}$$

By induction, all summands are elements of V_t and the proof is completed.

We consider $U(L) \otimes U(A)$ as a (right) module over $U(L) \otimes U(L)$, where the action is component-wise, the first component $U(L)$ acts via right multiplication and the second via the adjoint action of L on A. Remember that as A is abelian Lie algebra we have that the universal enveloping algebra $U(A)$ is isomorphic to the symmetric tensor algebra $S(A) = \bigoplus_{k\geq 1} S^k A$. The right action of $U(L)$ on $U(A)$ can be described as follows: for $w_1, \ldots, w_k \in A, l \in L$ the image of $w_1 \cdots w_k \in S^k A \subset U(A)$ under the action of l is $(w_1 \cdots w_k) \circ l = \sum_{1\leq i\leq k} w_1 \cdots (w_i \circ l) \ldots w_k$; note that in this case, as $l \in L$ we have $w_i \circ l = [w_i, l]$. We write $*$ for the described action of $U(L) \otimes U(L)$ on $U(L) \otimes U(A)$ and remind the reader that by definition for a subspace J of $U(F), U(Q)$, $U(F) \otimes U(F)$ or $U(Q) \otimes U(Q)$ the subspace J_t is spanned by all elements of J of degree at most t.

LEMMA 3.2: *Let μ: $U(L) \otimes U(A) \to U(L)$ be the linear map sending $\lambda_1 \otimes \lambda_2$ to $\lambda_1\lambda_2$. Then*

1. *for $i \leq m, j \leq s_0$ and $\lambda \in (\operatorname{Ker}\rho \otimes \rho)_{2t+1}$ we have*

$$e_i\mu((1 \otimes a_j) * (\varphi \otimes \varphi)(\lambda)) \in V_t;$$

2. *the map* μ: $U(L)\otimes U(A)\to U(L)$ *is a homomorphism of* $U(L)$*-modules where* $U(L)$ *acts diagonally on the domain, i.e., via the diagonal homomorphism* $U(L)\to U(L)\otimes U(L)$ *sending* $l\in L$ *to* $l\otimes 1+1\otimes l$.

Proof: Note that $\mathrm{Ker}(\rho\otimes\rho)_{2t+1}$ is spanned by $p\Delta q$, where p,q are monomials in $U(F)\otimes U(F)$, $\deg(pq)\le 2t-1$ and Δ is $[X_\alpha,X_\beta]\otimes 1$ or $1\otimes[X_\alpha,X_\beta]$ for some $\alpha>\beta$. For the subset $U(L)\otimes A$ of $U(L)\otimes U(A)$ we have

$$(U(L)\otimes A)*(\varphi\otimes\varphi)(1\otimes[X_\alpha,X_\beta])\subseteq U(L)\otimes(A\circ[y_\alpha,y_\beta])=0.$$

Hence it is sufficient to consider only the case $\Delta=[X_\alpha,X_\beta]\otimes 1$. We write $p=p_1(U)p_2(V), q=q_1(U)q_2(V)$ for some monomials $p_1,p_2,q_1,q_2\in U(F)$. Then $(1\otimes a_j)*(\varphi\otimes\varphi)(p\Delta q)=\varphi(p_1[X_\alpha,X_\beta]q_1)\otimes(a_j\circ\rho(p_2q_2))$ and using $[y_\alpha,y_\beta]=a_{\alpha,\beta}\in\{a_1,\dots,a_{s_0}\}\cup\{0\}$ we get

$$\begin{aligned}\mu((1\otimes a_j)*(\varphi\otimes\varphi)(p\Delta q))&=\varphi(p_1)a_{\alpha,\beta}\varphi(q_1)(a_j\circ\rho(p_2q_2))\\ =\varphi(p_1)[a_{\alpha,\beta},\varphi(q_1)](a_j\circ\rho(p_2q_2))&+\varphi(p_1)\varphi(q_1)a_{\alpha,\beta}(a_j\circ\rho(p_2q_2))\\ =\varphi(p_1)(a_{\alpha,\beta}\circ\rho(q_1))(a_j\circ\rho(p_2q_2))&+\varphi(p_1q_1)a_{\alpha,\beta}(a_j\circ\rho(p_2q_2)).\end{aligned}$$

By Lemma 3.1 both summands are in V_t.

The second part of the lemma follows immediately from the definition of the map μ.

PROPOSITION 3.3: *There exists a positive integer* t_0 *such that for* $t\ge t_0$ *we have* $V_t=V_{t+1}$.

Proof: We fix $t_0=\max\{ln,e_0-l-1\}$, where l is the positive integer used in (2) and e_0 is the maximal degree of a monomial in (2).

Let f_1,f_2 be monomials in $U(F)$ with $\deg(f_1f_2)=t+1\ge t_0+1$. If $f_1\ne 1$ we write $f_1=gY$ for some $Y\in\{X_1,\dots,X_n\}$. Then

$$\begin{aligned}\varphi(f_1)(a_j\circ\rho(f_2))=\varphi(g)\varphi(Y)(a_j\circ\rho(f_2))&=\varphi(g)[\varphi(Y),a_j\circ\rho(f_2)]\\ +\varphi(g)(a_j\circ\rho(f_2))\varphi(Y)=-\varphi(g)(a_j\circ\rho(f_2Y))&+\varphi(g)(a_j\circ\rho(f_2))\varphi(Y),\end{aligned}$$

i.e., $e_i\varphi(f_1)(a_j\circ\rho(f_2))$ is in the $U(L)$-submodule generated by the elements $e_i\varphi(f)(a_j\circ\rho(f_3))$ for $\deg(f)<\deg(f_1),\deg(ff_3)\le\deg(f_1f_2)$. Therefore to complete the proof of the proposition it is sufficient to show $e_i(a_j\circ\rho(f))\in V_t$ for all monomials f in $U(F)$ with $\deg(f)=t+1$.

As $t+1\ge ln+1$ we can assume that for some k we have $\alpha_k\ge l+1$ and $f=X_k^{l+1}X_1^{\alpha_1}\cdots X_{k-1}^{\alpha_{k-1}}X_k^{\alpha_k-l-1}X_{k+1}^{\alpha_{k+1}}\cdots X_n^{\alpha_n}$ (remember $\rho(f)\in U(Q)$ and $U(Q)$ is commutative). Then (2) implies

$$(x_k^{l+1}x_1^{\alpha_1}\cdots x_{k-1}^{\alpha_{k-1}}x_k^{\alpha_k-l-1}x_{k+1}^{\alpha_{k+1}}\cdots x_n^{\alpha_n})(v)+\alpha+\beta+\gamma=0, \tag{3}$$

where

$$\alpha=\sum_{i\le l}x_k(v)^i f_{rski}(d)(x_1^{\alpha_1}\cdots x_{k-1}^{\alpha_{k-1}}x_k^{\alpha_k-l-1}x_{k+1}^{\alpha_{k+1}}\cdots x_n^{\alpha_n})(v),$$
$$\beta=\sum_j g_{rj}(u)\phi_{rskj}(x_1^{\alpha_1}\cdots x_{k-1}^{\alpha_{k-1}}x_k^{\alpha_k-l-1}x_{k+1}^{\alpha_{k+1}}\cdots x_n^{\alpha_n})(v),$$
$$\gamma=\sum_j \widetilde{g}_{sj}(v)\psi_{rskj}(x_1^{\alpha_1}\cdots x_{k-1}^{\alpha_{k-1}}x_k^{\alpha_k-l-1}x_{k+1}^{\alpha_{k+1}}\cdots x_n^{\alpha_n})(v).$$

The degrees of the elements involved in (3) are bounded above by $e_0+\deg(f)-l-1=e_0+t-l\le 2t+1$.

Note that α belongs to the $U(Q)$-submodule of $U(Q)\otimes U(Q)$ (via the diagonal action) generated by the subspace $(U(Q)\otimes U(Q))_t$. We can lift α to an element $\widetilde{\alpha}$ from the $U(F)$-submodule of $U(F)\otimes U(F)$ generated by $(U(F)\otimes U(F))_t$, i.e., $(\rho\otimes\rho)(\widetilde{\alpha})=\alpha$. We can find $\widetilde{\beta}=\sum_j(\nu(g_{rj}))(U)\widetilde{\beta}_j$, $\widetilde{\gamma}=\sum_j(\nu(\widetilde{g}_{sj}))(V)\widetilde{\gamma}_j$ both in $(U(F)\otimes U(F))_{2t+1}$ such that $(\rho\otimes\rho)(\widetilde{\beta})=\beta,(\rho\otimes\rho)(\widetilde{\gamma})=\gamma$. We remind the reader that for an element $\lambda\in U(F)$ the elements $\lambda(U)$, $\lambda(V)$ and $\lambda(D)$ in $U(F)\otimes U(F)$ are $\lambda\otimes 1, 1\otimes\lambda$ and the image of λ under the diagonal homomorphism $U(F)\to U(F)\otimes U(F)$. Then (3) implies

$$f(V)+\widetilde{\alpha}+\widetilde{\beta}+\widetilde{\gamma}\in \operatorname{Ker}(\rho\otimes\rho)_{2t+1}. \tag{4}$$

Now Proposition 3.3 follows from Lemma 3.4. Indeed, Lemma 3.4 together with (4) implies $e_r(a_s\circ\rho(f))=e_r\mu((1\otimes a_s)*(\varphi\otimes\varphi)(f(V)))\in V_t$.

Lemma 3.4: *For $\lambda\in\{\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma}\}\cup\operatorname{Ker}(\rho\otimes\rho)_{2t+1}$,*

$$e_r\mu((1\otimes a_s)*(\varphi\otimes\varphi)(\lambda))\in V_t.$$

Proof: If $\lambda=\widetilde{\beta}$ then $e_r\mu((1\otimes a_s)*(\varphi\otimes\varphi)(\lambda))\in\sum_j e_r(\varphi\nu(g_{rj}))U(L)\subseteq\sum_{x\in\widetilde{X}}xU(L)\subseteq V_t$.

If $\lambda=\widetilde{\gamma}$ then $(1\otimes a_s)*(\varphi\otimes\varphi)(\lambda)=0$.

If $\lambda=\widetilde{\alpha}$ we use Lemma 3.2(2) to deduce $e_r\mu((1\otimes a_s)*(\varphi\otimes\varphi)(\lambda))\subseteq e_r\mu((1\otimes a_s)*(U(L)\otimes U(L))_t)U(L)\subseteq V_t$.

Finally, if $\lambda\in\operatorname{Ker}(\rho\otimes\rho)_{2t+1}$ we use Lemma 3.2(1). This completes the proof of Lemma 3.4, Proposition 3.3 and Proposition 3.

3. Proofs of the main theorems

LEMMA 4: *Under the conditions of Theorem A, if B is finitely presented over $U(L)$ then $B \otimes A$ is finitely generated over $U(Q)$ via the diagonal action.*

Proof: Consider the following diagram with the first row an exact complex of $U(L)$-modules and the second row an exact complex of $U(Q)$-modules,

$$(5)\qquad \begin{array}{ccccccc} R_1 & \xrightarrow{\partial_1} & R_0 = \bigoplus_{i\le m} e_i U(L) & \xrightarrow{\partial_0} & B & \to & 0 \\ & & & & \downarrow 1_B & & \\ Q_1 = B\otimes A\otimes U(A) & \xrightarrow{d_1} & Q_0 = B\otimes U(A) & \xrightarrow{d_0} & B & \to & 0 \end{array}$$

where R_0, R_1 are free $U(L)$-modules of finite rank, $\partial_0(e_i) = b_i, d_0(b\otimes\lambda) = b\epsilon(\lambda)$, ϵ is the augmentation map $U(A) \to K$ and $d_1(b\otimes a\otimes\lambda) = b\otimes a\lambda$. Note that the bottom row is obtained by tensoring a beginning of the standard resolution for K over $U(A)$ with the module B. As all our tensors are over a field tensoring preserves the exactness, i.e., the bottom row is exact in a sense that $\operatorname{Im} d_1 = \operatorname{Ker} d_0$ and d_0 is surjective. Note we do not claim that d_1 is injective, in fact this is not true. The same remark is valid for the top row, i.e., in general ∂_1 is not injective but $\operatorname{Im}\partial_1 = \operatorname{Ker}\partial_0$, ∂_0 is surjective.

Let $\alpha\colon U(Q) \to U(L)$ be the composition $\varphi\circ\nu$, where φ and ν are the maps defined in section 2. We fix a finite generating set $\{\sum_i e_i\lambda_{i,j}\}_j \subset \bigoplus_{i\le m} e_i U(Q)$ over $U(Q)$ of the kernel of the $U(Q)$-homomorphism $\bigoplus_{i\le m} e_i U(Q) \to B$ sending e_i to b_i. Then

$$\operatorname{Ker}\partial_0 = \sum_{i\le m}(e_iA)U(L) + \sum_j\Big(\sum_{i\le m} e_i\alpha(\lambda_{i,j})\Big)U(L)$$

and we can assume R_1 has a finite basis $X_1 \cup X_2$ such that $\partial_1(X_1) \subseteq \bigcup_{i\le m} e_iA$, $X_2 = \{x_{2,j}\}_j, \partial_1(x_{2,j}) = \sum_{i\le m} e_i\alpha(\lambda_{i,j})$.

Now we want to construct homomorphisms of $U(A)$-modules $\beta_i\colon R_i \to Q_i$ for $i = 0,1$ that extend the identity on B and commute with the differential of the diagram (5). Define $\beta_0\colon R_0 \to Q_0$ by

$$\beta_0(e_i\alpha(x_1^{k_1}\cdots x_n^{k_n})\lambda) = (b_i(x_1^{k_1}\cdots x_n^{k_n}))\otimes\lambda \text{ for } \lambda\in U(A).$$

We view $B\otimes U(A)$ as a $U(Q)\otimes U(Q)$-module, where for $b\in B, \lambda\in U(A), t_1, t_2 \in U(Q)$ we have that the image $(b\otimes\lambda)\hat{*}(t_1\otimes t_2)$ of $b\otimes\lambda$ under $t_1\otimes t_2$ is $bt_1\otimes(\lambda\circ t_2)$, where $\circ$ is the action of $U(Q)$ on $U(A)$ defined in the previous section. The definition of β_1 is as follows: $\beta_1(X_2U(L)) = 0$ and for $x\in X_1, \lambda\in U(A)$ such that $\partial_1(x) = e_ia$

$$\beta_1(x\alpha(x_1^{k_1}\cdots x_n^{k_n})\lambda) = ((b_i\otimes a)\hat{*}\delta(x_1^{k_1}\cdots x_n^{k_n}))\otimes\lambda,$$

where $\delta\colon U(Q) \to U(Q) \otimes U(Q)$ is the diagonal homomorphism.

Now we show that $d_1\beta_1 = \beta_0\partial_1$. Note that it is sufficient to show $d_1\beta_1(xl) = \beta_0\partial_1(xl)$, where $x \in X, l = \alpha(x_1^{k_1} \cdots x_n^{k_n})\lambda \in U(L)$ for $\lambda \in U(A)$. Suppose first that $x = x_{2,j} \in X_2$. Then $\beta_1(xl) = 0$ and $\beta_0\partial_1(xl)$ is

$$\beta_0(\sum_i e_i\alpha(\lambda_{i,j})\alpha(x_1^{k_1} \cdots x_n^{k_n})\lambda) = (\sum_i b_i\lambda_{i,j}x_1^{k_1} \cdots x_n^{k_n}) \otimes \lambda = 0 \otimes \lambda = 0.$$

Hence $d_1\beta_1(xl) = 0 = \beta_0\partial_1(xl)$, as required.

Now we suppose that $x \in X_1$ and $\partial_1(x) = e_ia$. Then $d_1\beta_1(xl) = d_1((b_i \otimes a)\hat{*}\partial(x_1^{k_1} \ldots x_n^{k_n}) \otimes \lambda)$ and $\beta_0\partial_1(xl) = \beta_0(e_ia\alpha(x_1^{k_1} \cdots x_n^{k_n})\lambda)$. The following claim finishes off the proof that $d_1\beta_1(xl) = \beta_0\partial_1(xl)$.

Claim: *For every $b \in B, a \in A, \lambda \in U(A), r \in \alpha(U(Q))$ such that the image of e_ir in B is b, i.e., $\partial_1(e_ir) = b$, we have*

$$d_1((b \otimes a)\hat{*}\delta(x_1^{k_1} \cdots x_n^{k_n}) \otimes \lambda) = \beta_0(e_ira\alpha(x_1^{k_1} \cdots x_n^{k_n})\lambda).$$

Proof: We use induction on the sum of the absolute values of all k_i. In the case when all $k_i = 0$ we have $d_1(b \otimes a \otimes \lambda) = b \otimes a\lambda = \beta_0(e_ira\lambda)$.

Suppose $k_1 = \cdots = k_{i-1} = 0$, $k_i \neq 0$ and ϵ_i is the sign of k_i. Then $(b \otimes a)\hat{*}\delta(x_1^{k_1} \cdots x_n^{k_n})$ is

$$((b \otimes a)\hat{*}\delta(x_i^{\epsilon_i}))\hat{*}\delta(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n})) = (bx_i^{\epsilon_i} \otimes a + b \otimes (a \circ x_i^{\epsilon_i}))\hat{*}\delta(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n})$$

and $(e_ira\alpha(x_1^{k_1} \cdots x_n^{k_n}))$ is

$$e_ira\alpha(x_i^{\epsilon_i})\alpha(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n}) = (e_ir(a \circ x_i^{\epsilon_i}) + e_ir\alpha(x_i^{\epsilon_i})a)\alpha(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n}).$$

Then by the inductive assumption we have

$$\begin{aligned}
d_1((b \otimes a)\hat{*}\delta(x_1^{k_1} \cdots x_n^{k_n}) \otimes \lambda) &= d_1((bx_i^{\epsilon_i} \otimes a + b \otimes (a \circ x_i^{\epsilon_i}))\hat{*}\delta(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n}) \otimes \lambda) \\
&= d_1((bx_i^{\epsilon_i} \otimes a)\hat{*}\delta(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n}) \otimes \lambda) + d_1(b \otimes (a \circ x_i^{\epsilon_i}))\hat{*}\delta(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n}) \otimes \lambda) \\
&= \beta_0(e_iry_i^{\epsilon_i}a\alpha(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n})\lambda) + \beta_0(e_ir(a \circ x_i^{\epsilon_i})\alpha(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n})\lambda) \\
&= \beta_0((e_iry_i^{\epsilon_i}a + e_ir(a \circ x_i^{\epsilon_i}))\alpha(x_i^{k_i-\epsilon_i} \cdots x_n^{k_n})\lambda) = \beta_0(e_ira\alpha(x_1^{k_1} \cdots x_n^{k_n})\lambda),
\end{aligned}$$

as required. Note this completes the induction and the proof of the claim.

Now we extend the rows of the diagram (5) to projective resolutions $\mathcal{R}$ and $\mathcal{Q}$ over $U(L)$ and $U(A)$ respectively and extend β_0, β_1, to a chain map $\beta\colon \mathcal{R} \to \mathcal{Q}$ of complexes over $U(A)$. The resolution $\mathcal{Q}$ is chosen in a special way. By definition it is $B \otimes \mathcal{F}$ with differential the tensor product of the identity of B with the differential of $\mathcal{F}$, where $\mathcal{F}$ is the 'standard' resolution over $U(A)$

$$\mathcal{F}\colon \cdots F_i = \wedge^i A \otimes U(A) \to F_{i-1} = \wedge^{i-1} A \otimes U(A) \to \cdots \to F_0 = U(A) \to K \to 0$$

with differential sending

$$(a_1 \wedge \cdots \wedge a_i) \otimes \lambda \text{ to } \sum_j (-1)^j (a_1 \wedge \cdots \wedge \hat{a}_j \wedge \ldots \wedge a_i) \otimes a_j \lambda.$$

The complex $\mathcal{F}$ is exact by [C-E, Ch. 13, Thm 7.1]. Note that we can use both resolutions $\mathcal{R}$ and $\mathcal{Q}$ to calculate the abelian group $\mathrm{Tor}_i^{U(A)}(K,K)$. As the chain map β (over $U(A)$) extends the identity of K (i.e., $d_0\beta_0 = \partial_0$) we deduce that β induces an isomorphism between the homology groups $H_i(\mathcal{R} \otimes_{U(A)} K)$ and $H_i(\mathcal{Q} \otimes_{U(A)} K)$ (this is the same argument as the one that shows that the homology groups of a group or Lie algebra are independent of the resolutions chosen). As the differential of $\mathcal{Q}$ is the tensor product of the identity of B with the differential of $\mathcal{F}$ and all our tensors are over a field (the field K), we deduce that $H_i(\mathcal{Q} \otimes_{U(A)} K) \simeq B \otimes H_i(\mathcal{F} \otimes_{U(A)} K) \simeq B \otimes \wedge^i A$.

Finally note that $H_1(\mathcal{R} \otimes_{U(A)} K)$ is finitely generated over $U(Q)$. Then $B \otimes A \simeq H_1(\mathcal{Q} \otimes_{U(A)} K)$ is an $U(Q)$-module via β_1 and by the definition of β_1 the action of $U(Q)$ is the diagonal one. This completes the proof of Lemma 4.

Lemma 5: *If L is a split extension of A by Q and B is of homological type FP_m over $U(L)$ then $B \otimes (\wedge^m A)$ is finitely generated over $U(Q)$, where $U(Q)$ acts via the diagonal homomorphism $U(Q) \to^{\otimes^{m+1}} U(Q)$ sending $q \in Q$ to $\sum_{0 \le i \le m} \underbrace{1 \otimes \cdots \otimes 1}_{i \text{ times}} \otimes q \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-i \text{ times}}$.*

Proof: Suppose

$$\mathcal{R}\colon \cdots \to R_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} B \to 0$$

is a free resolution of B over $U(L)$ such that R_i for $i \le m$ is finitely generated and $\mathcal{Q} = B \otimes \mathcal{F}$ is the resolution considered in the proof of Lemma 4 with differentials denoted by d_i.

Now we construct a chain map $\alpha\colon \mathcal{R} \to \mathcal{Q}$ over $U(A)$ inducing identity on B. First $R_i = T_i \otimes_{U(A)} U(L) \simeq T_i \otimes_K U(Q)$ for some free $U(A)$-submodule T_i of R_i. We want to define α in such a way that $\alpha_i(tf) = \alpha_i(t)^f$ for all $t \in T_i$, f a monomial in $U(Q)$, where upper index f denotes the image under the diagonal action of f. We proceed by induction on i. Suppose we have constructed α_{i-1}; then there exists a homomorphism of $U(A)$-modules $\beta_i\colon T_i \to Q_i$ such that

$d_i\beta_i = \alpha_{i-1}\partial_i$. We set $\alpha_i(tf) = \beta_i(t)^f$ for $t \in T_i, f$ a monomial in $U(Q)$. It is easy to check that α_i is a homomorphism of $U(A)$-modules and $d_i\alpha_i = \alpha_{i-1}\partial_i$. Finally, α_i induces an isomorphism between the homology groups $H_i(\mathcal{Q}\otimes_{U(A)}K)$ and $H_i(\mathcal{R}\otimes_{U(A)}K)$. The latter is a finitely generated $U(Q)$-module for $i \leq m$ and by construction the induced by α action of $U(Q)$ on $H_i(\mathcal{Q}\otimes_{U(A)}K) \simeq B\otimes(\wedge^i A)$ is the diagonal one.

THEOREM 6: *Suppose A and B are finitely generated $U(Q)$-modules.*

1. *$B\otimes(\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action if and only if whenever $[v_2],\ldots,[v_{m+1}] \in \Delta(Q,A), [v_1] \in \Delta(Q,B)$ and $[0] = [v_1]+\cdots+[v_{m+1}]$ we have $[v_i] = [0]$ for all i.*
2. *If $B \otimes (\wedge^m A)$ is finitely generated over $U(Q)$ via the diagonal action, then $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action.*

Proof: 1. We write M for $B \otimes (\otimes^m A)$ and view it as a module over $\otimes^{m+1} U(Q)$. Then the diagonal embedding $\theta\colon U(Q) \to \otimes^{m+1} U(Q)$ induces a map

$$\theta^*\colon \Delta(Q^{m+1}, M) \to \Delta(Q, M).$$

By [B-G 2, Prop. 3.1] M is finitely generated over $U(Q)$ via the diagonal action if and only if $(\theta^*)^{-1}([0]) = [0]$. As shown in [B-G 2] there is a direct product formula

$$\Delta(Q^{m+1}, M) \simeq \Delta(Q,B) \times (\Delta(Q,A))^m$$

and under the identification given by the above isomorphism θ^* sends $([v_1],[v_2],\ldots,[v_{m+1}])$ to $\sum_j [v_j]$. This implies immediately the first part of the theorem.

2. Now we assume the second part of the theorem is wrong and then, by the first part, there exist $[v_2],\ldots,[v_{m+1}] \in \Delta(Q,A)$ not all $[0]$ and $[v_1] \in \Delta(Q,B)$ such that $[v_1] + \cdots + [v_{m+1}] = [0]$.

We apply Lemma 2 for the linear maps $\alpha_i = \mu_i v_i\colon Q \to \overline{K}((t_i))$, where $\mu_i\colon \overline{K}((t)) \to \overline{K}((t_i))$ is the isomorphism of $\overline{K}$-algebras sending t to t_i, and obtain non-trivial linear maps

$$w_1\colon B \to \overline{K}((t_1)), \quad w_i\colon A \to \overline{K}((t_i)) \quad \text{for all } 2 \leq i \leq m+1$$

with the properties described in Lemma 2, i.e., $w_1(bq) = w_1(b)v_1(q)$ and $w_i(aq) = w_i(a)v_i(q)$ for $2 \leq i \leq m+1$. Using the maps w_i we construct another linear map

$$\widetilde{w} = w_1 \otimes w_2 \otimes \cdots \otimes w_{m+1}\colon B \otimes (\otimes^m A) \to R = \overline{K}((t_1)) \otimes \overline{K}((t_2)) \otimes \cdots \otimes \overline{K}((t_{m+1}))$$

that will play an important role in the completion of the proof of Theorem 6.

Let

$$\alpha \colon B \otimes (\otimes^m A) \to B \otimes (\otimes^m A)$$

be the linear map given by $\alpha(b\otimes a_1 \otimes \cdots \otimes a_m) = \sum_{\sigma \in S_m} (-1)^\sigma b \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(m)}$. As the image of α factors through $B \otimes (\wedge^m A)$ it is finitely generated over $U(Q)$. Note that $\operatorname{Im} \alpha$ is a module over $U(Q) \otimes S$ and α is a homomorphism of $U(Q)\otimes S$-modules, where $S = \{\lambda \in^{\otimes^m} U(Q) \mid \lambda\sigma = \lambda \text{ for all } \sigma \in S_m\}$ and the symmetric group S_m acts on $\otimes^m U(Q)$ permuting the factors of the tensor product, i.e., σ sends $\lambda_1 \otimes \cdots \otimes \lambda_m$ to $\lambda_{\sigma(1)} \otimes \cdots \otimes \lambda_{\sigma(m)}$. An arbitrary element $t \in^{\otimes^m} U(Q)$ is a root of the polynomial $\prod_{\sigma \in S_m}(x - (t)\sigma) \in S[x]$ and hence t is integral over S. In addition, $\otimes^m U(Q)$ is a finitely generated commutative algebra over K (with minimal number of generators mn). Then $\otimes^m U(Q)$ is integral over S, the K-algebra $\otimes^{m+1} U(Q)$ is integral over $U(Q) \otimes S$ and so $V = \operatorname{Im} \alpha(\otimes^{m+1} U(Q))$ is finitely generated over $U(Q)$.

Now let s be the non-negative integer with the properties $\widetilde{w}(V) \subseteq J^s$ and $\widetilde{w}(V) \not\subseteq J^{s+1}$, where J is the ideal of R generated by $t_1 - t_2, t_2 - t_3, \ldots, t_m - t_{m+1}$. By definition, J^0 is R. Then for $v \in V$ the image of the diagonal action of $q \in Q$ on $\widetilde{w}(v)$ is $\widetilde{w}(v) \sum_i \alpha_i(q) \equiv \widetilde{w}(v) \sum_i \pi_i \alpha_i(q)$ modulo J^{s+1}, where $\pi_i \colon \overline{K}((t_i)) \to \overline{K}((t_1))$ is the isomorphism of $\overline{K}$-algebras sending t_i to t_1. As $\sum_i [v_i] = 0$ we have $\sum_i \pi_i \alpha_i(q) \in \overline{K}[[t_1]]$ and hence $(\widetilde{w}(V) + J^{s+1})/J^{s+1}$ lies in a finitely generated $\overline{K}[[t_1]]$-submodule of $J^s/J^{s+1} \simeq$ a finitely generated, free $\overline{K}((t_1))$-module.

Finally, we choose v_i and $q \in Q$ such that $\operatorname{Im} \alpha_i$ is not a subset of $\overline{K}[[t_i]]$ and $\alpha_i(q) \notin \overline{K}[[t_i]]$ and define $h = (\otimes^{i-1} 1) \otimes q \otimes (\otimes^{m-i+1} 1) \in^{\otimes^{m+1}} U(Q)$. Then for $v \in V$ we have $\widetilde{w}(vh) = \widetilde{w}(v)\alpha_i(q) \equiv \widetilde{w}(v)\pi_i(\alpha_i(q))$ modulo J^{s+1} and hence $\widetilde{w}(V) + J^{s+1}/J^{s+1}$ is invariant under multiplication with f^j for every $j \geq 1$ where $f = \pi_i(\alpha_i(q)) \in \overline{K}((t_1)) \setminus \overline{K}[[t_1]]$. In particular, $(\widetilde{w}(V) + J^{s+1})/J^{s+1}$ cannot lie in a finitely generated $\overline{K}[[t_1]]$-submodule of $J^s/J^{s+1} \simeq$ a finitely generated, free $\overline{K}((t_1))$-module, a contradiction.

THEOREM 7: *If A and B are finitely generated $U(Q)$-modules and $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action, then B is of type FP_m over $U(L)$, where the Lie algebra L is the split extension of A by Q.*

Proof: The proof of Theorem 7 is based on the existence of some special long exact sequences given by Lemma 7.1.

LEMMA 7.1: *For every $k \geq 1$ the complex*

$$0 \to \wedge^k A \xrightarrow{\partial_{k,k}} \cdots \xrightarrow{\partial_{i+1,k}} \wedge^i A \otimes S^{k-i} A \xrightarrow{\partial_{i,k}} \cdots \xrightarrow{\partial_{1,k}} S^k A \to 0$$

with differentials

$$\partial_{i,k}((a_1 \wedge \cdots \wedge a_i) \otimes (b_1 \otimes \cdots \otimes b_{k-i})) = \sum_{1 \leq j \leq i} (-1)^{i-j} (a_1 \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_i) \otimes (a_j \otimes b_1 \otimes \cdots \otimes b_{k-i})$$

for any $a_1, \ldots, a_i, b_1, \ldots, b_{k-i}$ *in* A, *is exact.*

Proof: Choose a basis A_0 of A and order it linearly. Then $\wedge^i A \otimes S^{k-i} A$ has a basis $\{(a_1 \wedge \cdots \wedge a_i) \otimes (b_1 \otimes \cdots \otimes b_{k-i}) \mid a_1, \ldots, a_i, b_1, \ldots, b_{k-i} \in A_0, a_1 < \cdots < a_i, b_1 \leq \cdots \leq b_{k-i}\} = \mathcal{X}_{i,k}$. We call an element of $\mathcal{X}_{i,k}$ good if $b_1 \geq a_1$ and define by $(\wedge^i A \otimes S^{k-i} A)_{good}$ the space spanned by the good elements. A partial order on $\mathcal{X}_{i,k}$ is defined by $(a_1 \wedge \cdots \wedge a_i) \otimes (b_1 \otimes \cdots \otimes b_{k-i}) \leq (a'_1 \wedge \cdots \wedge a'_i) \otimes (b'_1 \otimes \cdots \otimes b'_{k-i})$ if and only if $a_j \leq a'_j$ for all $j \leq i$.

CLAIM 7.1.1: $\wedge^i A \otimes S^{k-i} A = (\wedge^i A \otimes S^{k-i} A)_{good} + \operatorname{Im} \partial_{i+1,k}$

Proof: We show that a non-good element $(a_1 \wedge \cdots \wedge a_i) \otimes (b_1 \otimes \cdots \otimes b_{k-i})$ of $\mathcal{X}_{i,k}$ can be expressed modulo the image of $\partial_{i+1,k}$ as a sum of smaller elements of $\mathcal{X}_{i,k}$. Indeed

$$(a_1 \wedge \cdots \wedge a_i) \otimes (b_1 \otimes \cdots \otimes b_{k-i}) + (-1)^{i+1} \partial_{i+1,k} (b_1 \wedge a_1 \wedge \cdots \wedge a_i) \otimes (b_2 \otimes \cdots \otimes b_{k-i})$$

is a sum of elements of $\mathcal{X}_{i,k}$ smaller than $(a_1 \wedge \cdots \wedge a_i) \otimes (b_1 \otimes \cdots \otimes b_{k-i})$. This completes the proof of the claim.

It follows immediately from Claim 7.7.1 that

$$\wedge^i A \otimes S^{k-i} A = (\wedge^i A \otimes S^{k-i} A)_{good} + \partial_{i+1,k}((\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}). \tag{6}$$

We claim that the sum in (6) is direct and

$$\partial_{i+1,k}((\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}) \simeq (\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}.$$

For both statements it is sufficient to consider the case when A is finite dimensional. In this case we define $\mu(i,k)$ to be the dimension of $(\wedge^i A \otimes S^{k-i} A)_{good}$, i.e., the number of good elements in $\mathcal{X}_{i,k}$.

CLAIM 7.7.2: $\dim_K(\wedge^i A \otimes S^{k-i} A) = \mu(i,k) + \mu(i+1,k)$.

Proof: Note that the dimension of $\wedge^i A \otimes S^{k-i} A$ is the cardinality of $\mathcal{X}_{i,k}$. It remains to show that $\mu(i+1,k)$ is the number of non-good elements in $\mathcal{X}_{i,k}$. This can be done by showing a bijection between the non-good elements in $\mathcal{X}_{i,k}$ and the good elements of $\mathcal{X}_{i+1,k}$. If $(a_1 \wedge \cdots \wedge a_i) \otimes (b_1 \otimes \cdots \otimes b_{k-i})$ is a non-good element from $\mathcal{X}_{i,k}$, then $(b_1 \wedge a_1 \wedge \cdots \otimes a_i) \otimes (b_2 \otimes \cdots \otimes b_{k-i})$ is a good element of $\mathcal{X}_{i+1,k}$. The inverse holds too and the proof of Claim 7.7.2 is completed.

Note that Claim 7.7.2 together with (6) shows that

$$\wedge^i A \otimes S^{k-i} A = (\wedge^i A \otimes S^{k-i} A)_{good} \oplus \partial_{i+1,k}((\wedge^{i+1} A \otimes S^{k-i-1} A)_{good})$$

and that the restriction of $\partial_{i+1,k}$ on $(\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}$ is injective. Similarly, the restriction of $\partial_{i,k}$ on $(\wedge^i A \otimes S^{k-i} A)_{good}$ is injective and hence $\operatorname{Im} \partial_{i+1,k} = \operatorname{Ker} \partial_{i,k}$. This completes the proof of Lemma 7.1.

Now we define V_i for $i \geq 1$ to be the subspace of $\otimes^i A$ generated by the elements $\sum_{\sigma \in S_i} (-1)^\sigma \ a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)}$ for all $a_1, \ldots, a_i \in A$. Let W_i be the $U(A)$-submodule of $\otimes^{i-1} A \otimes U(A)$ generated by $V_i \subseteq (\otimes^{i-1} A) \otimes A \subset (\otimes^{i-1} A) \otimes U(A)$.

LEMMA 7.2: *The map φ_i: $V_i \otimes U(A) \to W_i$ sending $\sum_{\sigma \in S_i} (-1)^\sigma a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)} \otimes \lambda$ to $\sum_{\sigma \in S_i} (-1)^\sigma a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i-1)} \otimes a_{\sigma(i)} \lambda$ has kernel W_{i+1}.*

Proof: We identify V_i with $\wedge^i A$ via the linear map θ_i: $\wedge^i A \to V_i$ sending $a_1 \wedge \cdots \wedge a_i$ to $\sum_{\sigma \in S_i} (-1)^\sigma \ a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)}$. Write $U(A)$ as a direct sum of the symmetric powers of A. We show that $(\theta_{i-1}^{-1} \otimes \mathrm{id}_{S^{k-i+1} A}) \varphi_i (\theta_i \otimes \mathrm{id}_{S^{k-i} A})$: $\wedge^i A \otimes S^{k-i} A \to \wedge^{i-1} A \otimes S^{k+1-i} A$ is the map $\partial_{i,k}$ defined in Lemma 7.1. Then Lemma 7.2 follows from Lemma 7.1.

Indeed for $j = k-i+1$, $(\theta_{i-1}^{-1} \otimes \mathrm{id}_{S^{k-i+1} A}) \varphi_i (\theta_i \otimes \mathrm{id}_{S^{k-i} A})(a_1 \wedge \cdots \wedge a_i \otimes b_1 \cdots b_j)$ is

$$\begin{aligned}
&(\theta_{i-1}^{-1} \otimes \mathrm{id}_{S^{k-i+1} A}) \varphi_i \Big(\sum_{\sigma \in S_i} (-1)^\sigma a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)} \otimes b_1 \cdots b_j \Big) \\
&= (\theta_{i-1}^{-1} \otimes \mathrm{id}_{S^{k-i+1} A}) \Big(\sum_{\sigma \in S_i} (-1)^\sigma a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i-1)} \otimes a_{\sigma(i)} b_1 \cdots b_j \Big) \\
&= (\theta_{i-1}^{-1} \otimes \mathrm{id}_{S^{k-i+1} A}) \Big(\sum_{1 \leq j \leq i} \ \sum_{\sigma \in S_i, \sigma(i) = j_0} (-1)^\sigma a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i-1)} \otimes a_{j_0} b_1 \cdots b_j \Big) \\
&= \sum_{j_0} (-1)^{i-j_0} (a_1 \wedge \cdots \hat{a}_{j_0} \cdots \wedge a_i) \otimes a_{j_0} b_1 \cdots b_j.
\end{aligned}$$

This completes the proof of the lemma.

LEMMA 7.3: *Under the assumptions of Theorem 7, for every $i \leq m$ the module $B \otimes W_i$ is of type FP_k over $U(L)$ if and only if $B \otimes W_{i+1}$ is of type FP_{k-1} over $U(L)$, where $U(A)$ acts on $B \otimes W_i$ via its action on the component W_i and $U(Q)$*

acts on $B \otimes (\otimes^{i-1} A) \otimes U(A)$ *via the diagonal map* $U(Q) \to^{\otimes^{i+1}} U(Q)$ *sending an element* q *from* Q *to* $\sum_{0 \leq j \leq i} \underbrace{1 \otimes \cdots \otimes 1}_{j \text{ times}} \otimes q \otimes \underbrace{1 \otimes \cdots \otimes 1}_{i-j \text{ times}}$.

Proof: The short exact sequence of $U(A)$-modules $0 \to W_{i+1} \to V_i \otimes U(A) \xrightarrow{\varphi_i} W_i \to 0$ gives rise to a short exact sequence of $U(L)$-modules

$$(7) \qquad 0 \to B \otimes W_{i+1} \to B \otimes V_i \otimes U(A) \xrightarrow{\mathrm{id}_B \otimes \varphi_i} B \otimes W_i \to 0,$$

where $U(Q)$ acts diagonally on all modules in (7). By Theorem 6(1), $B \otimes (\otimes^i A)$ is finitely generated over $U(Q)$ via the diagonal action for all $i \leq m$ and hence its submodule $B \otimes V_i$ is finitely generated over $U(Q)$. Then $(B \otimes V_i) \otimes U(A) \simeq (B \otimes V_i) \otimes_{U(Q)} U(L)$ is induced from a module of type FP_∞ over $U(Q)$ and is itself of type FP_∞ over $U(L)$. The dimension shifting argument [B, Prop 1.4] applied to (7) completes the proof.

Finally, we are ready to complete the proof of Theorem 7. Applying Lemma 7.3 several times we obtain $B \otimes W_1$ is of type FP_{m-1} over $U(L)$ if and only if $B \otimes W_m$ is of type FP_0 (i.e., finitely generated) over $U(L)$. Note that $B \otimes V_m$ is a generating set of $B \otimes W_m$ over $U(A)$. By assumption, $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ and so $B \otimes V_m$ is finitely generated over $U(Q)$.

Finally, it remains to show that $B \otimes W_1$ is of type FP_{m-1} over $U(L)$ if and only if B is of type FP_m over $U(L)$. This follows immediately from dimension shifting argument for the short exact sequence of $U(L)$-modules

$$0 \to B \otimes W_1 \to B \otimes_K U(A) \simeq B \otimes_{U(Q)} U(L) \to B \to 0$$

induced from the short exact sequence $0 \to W_1 \to U(A) \to K \to 0$.

Proof of Theorem A: $1 \Leftrightarrow 2$ by Proposition 3 and Lemma 4, $2 \Leftrightarrow 3$ by Theorem 6(1).

Proof of Corollary B: It is a straight corollary of Theorem A and the classification of finitely presented Lie algebras in [B-G 1], [B-G 2].

Proof of Theorem C: The theorem follows from Lemma 5, Theorem 6 and Theorem 7.

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